Lecture 20: Spectral Sequence (I)

Many applications of (co)homology theory are reduced to the computation

$$\mathrm{H}(C^\bullet, \delta)$$

of (co)homologies of certain (co)chain complexes. Usually the differential δ is complicated, making the computation difficult.

However, if we observe "part" of the differential δ is simple, say

$$\delta = \delta_1 + \delta_2$$

while the computation of δ_1-cohomology is easier to perform, then we would like to use the δ_1-cohomology to compute the full δ-cohomology. This is the idea of spectral sequence.

Motivation

Let us motivate this idea by a standard example.

Consider the double complex

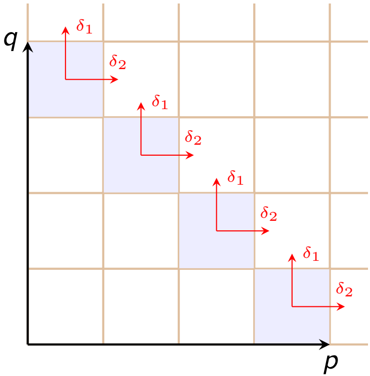
$$K = \bigoplus_{p,q \geq 0} K^{p,q}$$

which is equipped with two differentials

$$\begin{cases} \delta_1 : K^{p,q} \to K^{p,q+1} \\ \delta_2 : K^{p,q} \to K^{p+1,q} \end{cases}$$

such that

$$\delta_1^2 = \delta_2^2 = 0, \quad \delta_1\delta_2 + \delta_2\delta_1 = 0.$$



Consider the total complex

$$\mathrm{Tot}^\bullet(K), \quad \mathrm{Tot}^n(K) = \bigoplus_{p+q=n} K^{p,q}$$

with the differential

$$D = \delta_1 + \delta_2.$$

Our assumption on δ_1, δ_2 implies that

$$D^2 = 0.$$

Therefore $(\mathrm{Tot}^\bullet(K), D)$ indeed defines a cochain complex, and we are interested in

$$\mathrm{H}^\bullet(\mathrm{Tot}^\bullet(K), D).$$

Let x be a representative of an element in $\mathrm{H}^m(\mathrm{Tot}^\bullet(K), D)$. We can decompose x into

$$x = x_0 + x_1 + \cdots, \quad x_i \in K^{i,m-i}.$$

The cocycle condition $Dx = 0$ is equivalent to

$$\begin{cases} \delta_1 x_0 = 0 \\ \delta_2 x_0 = -\delta_1 x_1 \\ \delta_2 x_1 = -\delta_1 x_2 \\ \quad \vdots \end{cases}$$

Let us formally write

$$x_1 \text{``} = \text{''} -\delta_1^{-1}\delta_2 x_0, \quad x_2 \text{``} = \text{''} -\delta_1^{-1}\delta_2 x_1, \quad \cdots$$

Here the inverse δ_1^{-1} does not exist and this expression is only heuristic. Then we would solve

$$x \text{``} = \text{''} \frac{1}{1+\delta_1^{-1}\delta_2} x_0$$

while x_0 represents a cocycle for $(\mathrm{Tot}^\bullet(K), \delta_1)$.

Intuitively, we treat δ_2 as a perturbation of δ_1 and

$$D = (\delta_1 + \delta_2)``='' \delta_1(1 + \delta_1^{-1}\delta_2).$$

So

$$Dx``='' \delta_1(1 + \delta_1^{-1}\delta_2)\frac{1}{1 + \delta_1^{-1}\delta_2}x_0``='' \delta_1 x_0 = 0.$$

The above discussion is of course vague and heuristic. But it motivates the following idea: we can construct a D-cocycle x by first looking at a δ_1-cocycle x_0 as a leading approximation, and then constructing

$$x_1, x_2, \cdots$$

order by order using information from $\mathrm{H}^\bullet(\delta_1)$.

This leads to the following statements.

If $\mathrm{H}^\bullet(\delta_1) = 0$, then $\mathrm{H}^\bullet(D) = 0$.

In fact, let x be a D-cocyle as above. Since

$$\delta_1 x_0 = 0$$

and $\mathrm{H}^\bullet(\delta_1) = 0$, we can find

$$y_0 \in K^{0,m-1} \quad \text{such that} \quad x_0 = \delta_1 y_0.$$

Replacing x by $x - Dy_0$, we can assume $x_0 = 0$ so x starts from x_1.

Then

$$Dx = 0 \Rightarrow \delta_1 x_1 = 0.$$

By the same reason, we can further kill x_1 to assume that x starts from x_2. Iterating this process, we can eventually find y such that

$$x = Dy.$$

So x is a D-coboundary. It follows that $\mathrm{H}^\bullet(D) = 0$.

If $\mathrm{H}^\bullet(\delta_1) \neq 0$, then we need to understand

whether $\delta_1 x_{i+1} = -\delta_2 x_i$ is solvable.

This puts extra condition on the initial data x_0 that allows to be an approximation of a D-cocycle. For example, we want to solve

$$\delta_1 x_1 = -\delta_2 x_0.$$

Since

$$\delta_1(\delta_2 x_0) = -\delta_2 \delta_1 x_0 = 0,$$

we know $-\delta_2 x_0$ is δ_1-closed. The problem is

whether $-\delta_2 x_0$ is δ_1-exact.

We can view

$$\delta_2 : \mathrm{H}^\bullet(\delta_1) \to \mathrm{H}^\bullet(\delta_1)$$

as defining a cochain complex $(\mathrm{H}^\bullet(\delta_1), \delta_2)$, then the solvability of x_1 asks that the class $[x_0] \in \mathrm{H}^\bullet(\delta_1)$ is in fact δ_2-closed

$$\delta_2[x_0] = 0.$$

Therefore the "2nd"-order approximation of a D-cohomolpgy is

$$\mathrm{H}^\bullet(\mathrm{H}^\bullet(\delta_1), \delta_2).$$

This will be called the E_2-page. Similarly, we will have E_3-page, E_4-page, etc, and eventually the full description of D-cohomologies. Such process is the basic idea of spectral sequence.

Spectral sequence for filtered chain complex

Spectral sequences usually arise in two situations

1. A $\mathbb{Z}$-filtration of a chain complex: a sequence of subcomplexes $\cdots \subset F_p \subset F_{p+1} \subset \cdots$.
2. A $\mathbb{Z}$-filtration of a topological space: a family of subspaces $\cdots \subset X_p \subset X_{p+1} \subset \cdots$.

We first discuss the spectral sequence for chain complexes.

Definition

An (ascending) filtration of an R-module A is an increasing sequence of submodules

$$\cdots \subset F_pA \subset F_{p+1}A \subset \cdots$$

indexed by $p \in \mathbb{Z}$. We always assume it is exhaustive and Hausdorff

$$\bigcup_p F_pA = A \quad \text{(exhaustive)}, \qquad \bigcap_p F_pA = 0 \quad \text{(Hausdorff)}.$$

The filtration is bounded if $F_pA = 0$ for p sufficiently small and $F_pA = A$ for p sufficiently large.

The associated graded module $\mathrm{Gr}^F_\bullet A$ is defined by

$$\mathrm{Gr}^F_\bullet(A) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Gr}^F_p A, \quad \mathrm{Gr}^F_p A := F_pA/F_{p-1}A.$$

Definition

A filtered chain complex is a chain complex $(C_\bullet, \partial)$ together with an (ascending) filtration F_pC_i of each C_i such that the differential preserves the filtration

$$\partial(F_pC_i) \subset F_pC_{i-1}.$$

In other words, we have an increasing sequence of subcomplexes

$$F_pC_\bullet \subset C_\bullet.$$

Remark

There is also the notion of a descending filtration. We will focus on the ascending case here.

A filtered chain complex induces a filtration on its homology

$$F_p \mathrm{H}_i(C_\bullet) = \mathrm{Im}(\mathrm{H}_i(F_p C_\bullet) \to \mathrm{H}_i(C_\bullet)).$$

In other words, an element $[\alpha] \in \mathrm{H}_i(C_\bullet)$ lies in $F_p \mathrm{H}_i(C_\bullet)$ if and only if there exists a representative $x \in F_p C_i$ such that $[\alpha] = [x]$.

Its graded piece is given by

$$\mathrm{Gr}_p^F \mathrm{H}_i(C_\bullet) = \frac{\mathrm{Ker}(\partial : F_p C_i \to F_p C_{i-1})}{F_{p-1} C_i + \partial C_{i+1}}.$$

Notation

Our notation of quotient means the quotient of the numerator by its intersection with the denominator, i.e., $\frac{A}{B} := \frac{A}{A \cap B}$.

Definition

Given a filtered R-module A, we define its Rees module as a submodule of $A[z, z^{-1}]$ by

$$A_F := \bigoplus_{p\in\mathbb{Z}} F_pA\, z^p \subset A[z, z^{-1}].$$

Our conditions for the filtration can be interpreted as follows

1. increasing filtration: A_F is a $R[z]$-submodule of $A[z, z^{-1}]$ and $z : A_F \to A_F$ is injective.
2. exhaustive: $A_F[z^{-1}] := A_F \otimes_{R[z]} R[z, z^{-1}]$ equals $A[z, z^{-1}]$.
3. Hausdorff: $\bigcap_{p \geq 0} z^p A_F = 0$ in $A[z, z^{-1}]$.

The associated graded module is given by

$$\mathrm{Gr}^F_\bullet(A) := A_F / z A_F.$$

Geometrically, we can think about $A[z, z^{-1}]$ as the space of algebraic sections of the trivial bundle on $\mathbb{C}^*$ with fiber A.

Then A_F defines the extension of this bundle to $\mathbb{C}$, whose fiber at 0 is precisely $\mathrm{Gr}_\bullet^F(A)$.

Let $(C_\bullet, \partial, F_\bullet)$ be a filtered chain complex. Let us denote its Rees module by

$$C_F := \bigoplus_{p \in \mathbb{Z}} F_p C_\bullet \; z^p \subset C_\bullet[z, z^{-1}].$$

(C_F, ∂) is also a subcomplex of $(C_\bullet[z, z^{-1}], \partial)$. This defines a map

$$\mathrm{H}_\bullet(C_F, \partial) \to \mathrm{H}_\bullet(C_\bullet[z, z^{-1}], \partial) = \mathrm{H}_\bullet(C_\bullet, \partial)[z, z^{-1}].$$

The image of

$$\mathrm{H}_\bullet(C_F, \partial) \to \mathrm{H}_\bullet(C_\bullet, \partial)[z, z^{-1}].$$

defines a $\mathbb{C}[z]$-submodule of $\mathrm{H}_\bullet(C_\bullet, \partial)[z, z^{-1}]$. It induces a filtration on $\mathrm{H}_\bullet(C_\bullet, \partial)$ as described above.

Our goal is to analyze this map in order to extract the information about this induced filtration on $\mathrm{H}_\bullet(C_\bullet, \partial)$.

Firstly

$$\mathrm{H}_\bullet(C_F, \partial) = \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_\bullet(F_p C_\bullet, \partial) z^p.$$

However, the z-action

$$z : \mathrm{H}_\bullet(C_F, \partial) \to \mathrm{H}_\bullet(C_F, \partial)$$

may not be injective. Those elements that are annihilated by z^m for some finite m will be killed under φ.

One way to kill such elements is to look at $\mathrm{im}(z^N)$ for N big enough. This motivates the following construction.

Let us define

$$E^r := \frac{\{x \in C_F | \partial x \in z^r C_F\}}{zC_F + z^{1-r}\partial C_F}.$$

E^r can be viewed as the r-th order approximation. E^r carries a differential

$$\partial_r : E^r \to E^r, \quad [x] \to z^{-r}[\partial x].$$

Obviously, $\partial_r^2 = 0$. We can define its homology by

$$\mathrm{H}(E^r, \partial_r) := \frac{\ker \partial_r}{\operatorname{im} \partial_r}.$$

Claim

The homology of (E^r, ∂_r) is precisely E^{r+1}

$$\mathrm{H}(E^r, \partial_r) = E^{r+1}.$$

Proof: Assume $[x] \in \ker \partial_r$ in E^r. $\partial_r[x] = z^{-r}[\partial x] = 0$ implies the existence $\alpha, \beta \in C_F$ such that

$$\partial x = z^r(z\alpha + z^{1-r}\partial\beta) = z^{r+1}\alpha + z\partial\beta, \quad \partial\beta \in z^{r-1}C_F.$$

We have $\partial(x - z\beta) = z^{1+r}\alpha$, so $[x - z\beta]$ defines an element in E^{r+1}. This class does not depend on the choice of α, β.

Therefore we have a natural map

$$f\colon \ker \partial_r \to E^{r+1}$$

which is clearly surjective.

Assume $[x] = \partial_r[y]$. Then there exists $u, v \in C_F$ such that

$$x = z^{-r}\partial y + zu + z^{1-r}\partial v.$$

So

$$f([x]) = [x - zu] = [z^{-r}\partial(y + zv)] = 0.$$

Therefore

$$\operatorname{im} \partial_r \subset \ker f.$$

On the other hand, assume $f([x]) = 0$. Then there exists $u, v \in C_F$ such that

$$x - z\beta = zu + z^{-r}\partial v, \quad \partial u = z^r \alpha.$$

We find $[x] = \partial_r[v]$. Hence

$$\ker f \subset \operatorname{im} \partial_r.$$

It follows that $\ker f = \operatorname{im} \partial_r$. This proves the claim. $\square$

We can describe (E^r, ∂_r) explicitly in terms of components. Let

$$(C_F)_{p,q} := F_p C_{p+q}.$$

There is a natural identification

$$C_F = \bigoplus_{p,q \in \mathbb{Z}} (C_F)_{p,q}.$$

Similarly, we can decompose

$$E^r = \bigoplus_{p,q\in\mathbb{Z}} E^r_{p,q}$$

where

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} | \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial F_{p+r-1} C_{p+q+1}}.$$

The differential ∂_r acts on components by

$$\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}, \quad x \to \partial x.$$

E^0 is given by

$$E^0 = C_F/zC_f, \quad E^0_{p,q} = \mathrm{Gr}^F_p C_{p+q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}.$$

E^1 is given by

$$E^1 = \frac{\{x \in C_F | \partial x \in zC_F\}}{zC_F + \partial C_F} = \mathrm{H}(C_F/zC_F, \partial), \quad E^1_{p,q} = \mathrm{H}_{p+q}(\mathrm{Gr}^F_p C_\bullet).$$

If the filtration of C_i is bounded for each i, then for any p, q,

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} | \partial x = 0\}}{F_{p-1} C_{p+q} + \partial C_{p+q+1}} = \mathrm{Gr}_p \mathrm{H}_{p+q}(C_\bullet) \quad \text{for} \quad r >> 0.$$

In this case, we say $\{E^r\}_r$ converges to $\mathrm{Gr}_\bullet \mathrm{H}(C_\bullet)$ and write

$$E^\infty = \mathrm{Gr}_\bullet \mathrm{H}(C_\bullet).$$

Motivated by the above discussion, we now give the formal definition of spectral sequence.

Definition

A spectral sequence (of R-modules) consists of

- an R-module $E^r_{p,q}$ for any $p, q \in \mathbb{Z}$ and $r \geq 0$;
- a differential $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r+1}$ such that $\partial_r^2 = 0$ and $E^{r+1} = \mathrm{H}(E^r, \partial_r)$.

A spectral sequence converges if for any p, q, we have

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots \quad \text{for} \quad r >> 0.$$

This limit will be denoted by $E^\infty_{p,q}$.

Theorem

There is an associated spectral sequence for any filtered chain complex $(C_\bullet, \partial, F_\bullet)$ where

$$E^r_{p,q} = \frac{\{x \in F_pC_{p+q} | \partial x \in F_{p-r}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial F_{p+r-1}C_{p+q+1}}$$

and

$$\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}, \quad x \to \partial x.$$

The E^1-page of the spectral sequence is

$$E^1_{p,q} = \mathrm{H}_{p+q}(\mathrm{Gr}^F_p C_\bullet).$$

If the filtration of C_i is bounded for each i, then the spectral sequence converges and

$$E^\infty_{p,q} = \mathrm{Gr}_p \mathrm{H}_{p+q}(C_\bullet).$$